

M-branes on U-folds

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Abstract: We give a preliminary discussion of how the addition of extra coordinates in M-theory, which together with the original ones parametrise a U-fold, can serve as a tool for formulating brane dynamics with manifest U-duality. The redundant degrees of freedom are removed by generalised self-duality constraints or calibration conditions made possible by the algebraic structure of U-duality. This is the written version of an invited talk at the 7th International Workshop “Supersymmetries and Quantum Symmetries”, Dubna, July 30–August 4, 2007.

$D = 11$ supergravity, dimensionally reduced on a torus T^n , has a global symmetry $E_{n(n)}$. The scalar fields parametrise the coset $E_{n(n)}(\mathbb{Z}) \backslash E_{n(n)} / K(E_{n(n)})$ [1, 2, 3, 4]. The series has a natural continuation to the infinite-dimensional cases of E_9 , E_{10} and E_{11} , although it is clear that the content of the latter two is larger than the fields in the dimensionally reduced theory. They have been proposed to actually describe M-theory, either in a picture where space or space-time is emergent [5, 6, 7], or hypothetically already in $D = 11$ [8, 9].

With only gravity, the internal vielbein parametrises $SL(n, \mathbb{Z}) \backslash GL(n) / SO(n)$, the size and shape of the torus. The symmetry enhancement comes from “mixing” of gravitational and tensorial fields (dualised or not). On reduction to $d = 3$, even pure gravity gives a symmetry enhancement—the graviphotons are dualised to scalars and become part of a $SL(n + 1)$ “vielbein” (Ehlers symmetry) [10].

n	$E_{n(n)}$	$K(E_{n(n)})$
2	$SL(2) \times \mathbb{R}$	$SO(2)$
3	$SL(3) \times SL(2)$	$SO(3) \times SO(2)$
4	$SL(5)$	$SO(5)$
5	$Spin(5, 5)$	$(Spin(5) \times Spin(5)) / \mathbb{Z}_2$
6	$E_{6(6)}$	$USp(8) / \mathbb{Z}_2$
7	$E_{7(7)}$	$SU(8) / \mathbb{Z}_2$
8	$E_{8(8)}$	$Spin(16) / \mathbb{Z}_2$

Table 1. U-duality groups and their maximal compact subgroups.

Let us sketch how U-duality arises in a simple example, namely $n = 4$. We divide the 11-dimensional coordinates X^M in x^μ , $\mu = 1, \dots, 7$, coordinates on the uncompactified 7-dimensional space-time, and y^m , $m = 1, \dots, 4$, coordinates on the torus T^4 .

The massless bosonic fields are

$$\begin{aligned}
g_{\mu\nu}: & \quad \text{metric, singlet;} \\
C_{\mu\nu l} \leftrightarrow \tilde{C}_{\mu\nu}, C_{\mu\nu p}: & \quad \text{2-forms in } \mathbf{5} \text{ of } SL(5); \\
g_{\mu n}, C_{\mu np}: & \quad \text{1-forms in } \mathbf{10} \text{ of } SL(5); \\
g_{mn}, C_{mnp}: & \quad \text{scalars in } SL(5)/SO(5).
\end{aligned}$$

This matches with the decomposition of representations when $SL(5) \rightarrow SL(4) \times \mathbb{R}$:

$$\begin{aligned}
\begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \mathbf{5} & \rightarrow \mathbf{4}_{1/5} \oplus \mathbf{1}_{-4/5} \\
\begin{array}{|c|} \hline \square \\ \hline \end{array} \quad \mathbf{10} & \rightarrow \mathbf{6}_{2/5} \oplus \mathbf{4}_{-3/5} \\
\begin{array}{|c|} \hline \square \\ \square \\ \square \\ \hline \end{array} \quad \mathbf{24} & \rightarrow \mathbf{4}_1 \oplus (\mathbf{15} \oplus \mathbf{1})_0 \oplus \bar{\mathbf{4}}_{-1}
\end{aligned}$$

As we consider dimensional reduction, the fields do not depend on the coordinates of T^n .

The global symmetry $E_{n(n)}$ (*e.g.* $E_{5(5)} \approx SL(5)$) contains rigid transformations changing the shape and size of the torus as well as the C-field on the torus. This can be seen as diffeomorphisms and 2-form gauge transformations with parameters linear in coordinates. This suggests the enlargement of the set of coordinates to a “coordinate representation” of $E_{n(n)}$ [11, 12]. Note that this is the same representation as the set of massless 1-forms. The coordinate representation is given by the natural enlargement of the “graviphotons”.

The following table gives the coordinate representations case by case, together with their reduction to representations of $SL(n)$:

n	$E_{n(n)}$	repr.	P^m	Z_{mn}	$Z_{m_1\dots m_5}$	$Z_{m_1\dots m_6}$
2	$SL(2) \times \mathbb{R}$	$2 \oplus 1$	\times	\times		
3	$SL(3) \times SL(2)$	$(3, 2)$	\times	\times		
4	$SL(5)$	10	\times	\times		
5	$Spin(5, 5)$	16	\times	\times	\times	
6	$E_{6(6)}$	$27(\oplus 1)$	\times	\times	\times	(\times)
7	$E_{7(7)}$	56	\times	\times	\times	\times
8	$E_{8(8)}$	248	\times	\times	\times	\times +more...

Table 2. Coordinate representations.

Hull [11] realised that the enlarged internal spaces could be used to describe, and geometrize, classes of non-geometric solutions to M-theory. In a situation where the remaining space-time is topologically non-trivial, the complete space can be taken as a bundle of the extended internal space over space-time where the allowed holonomies are in the discrete U-duality group.

U-duality acts linearly on the coordinates of torus with extended coordinates, but not on T^n . (Consider *e.g.* a T-duality on T^1 with $R \leftrightarrow R^{-1}$. If one instead interchanges two circles with radii R and R^{-1} the patching is geometrical.)

The aim here is to initiate a search for formulations of brane dynamics using the extended coordinates. This gives a manifestly U-duality symmetric formulation of brane

dynamics, and can provide a geometric description of branes in situations that are not geometric in the unextended formalism. There must be constraints on the branes, so that the dependence on the extra coordinates is eliminated in a proper way. Note also that branes of different dimensionalities transform into each other under U-duality. This means that they are described by the same brane on the extended space, but some of its directions may be hidden in the extra coordinate directions.

Let us for a little while focus on and review the analogous questions for T-duality [13]. T-duality transformations form a subgroup $SO(n-1, n-1)$ of the U-duality group. It is a perturbative symmetry of string theory on T^n . (Probably inspired by Hitchin's generalised complex geometry,) Hull proposed that T^n should be enlarged to T^{2n} , on which T-duality acts linearly, in order to geometrize non-geometric solutions of string theory, where transition functions contain non-geometric T-duality transformations. He also gave a formulation of string dynamics on the extended space.

The metric and B -field parametrise an element of $SO(n, n)/(SO(n) \times SO(n))$. Let us call the vector index of $SO(n, n)$ (the tangent index) M and the flat vector indices of $SO(n) \times SO(n)$ a and a' .

$$\mathcal{E}_M^a = \begin{bmatrix} E^{ma} \\ F_m^a \end{bmatrix}, \quad \tilde{\mathcal{E}}_M^{a'} = \begin{bmatrix} \tilde{E}^{ma'} \\ \tilde{F}_m^{a'} \end{bmatrix},$$

$$\begin{aligned} E^{ma} &= \frac{1}{\sqrt{2}} e^{ma}, & \tilde{E}^{ma'} &= -\frac{1}{\sqrt{2}} e^{ma'}, \\ F_m^a &= \frac{1}{\sqrt{2}} (e_m^a - B_{mn} e^{na}), & \tilde{F}_m^{a'} &= \frac{1}{\sqrt{2}} (e_m^{a'} + B_{mn} e^{na'}). \end{aligned}$$

The invariant metrics are

$$L_{MN} = (\mathcal{E} \mathcal{E}^t - \tilde{\mathcal{E}} \tilde{\mathcal{E}}^t)_{MN} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$G_{MN} = (\mathcal{E} \mathcal{E}^t + \tilde{\mathcal{E}} \tilde{\mathcal{E}}^t)_{MN} = \begin{bmatrix} g^{mn} & g^{mp} B_{pn} \\ -B_{mp} g^{pn} & g_{mn} - B_{mp} g^{pq} B_{qn} \end{bmatrix}.$$

How is string dynamics realised on the extended space, the T-fold? We will give a somewhat simplified account, forgetting quantum issues. Let us for simplicity forget about the uncompactified directions (they are of course important, but the technical issue is to get rid of unwanted dependence of the “too many” compactified directions). It is straightforward to insert *e.g.* the graviphoton field later.

We call the coordinates $Z^M = (Y_m, X^m)$. The pullbacks of the frame 1-forms to the string world-sheet are

$$\begin{aligned} \Pi^a &= dX^m e_m^a + (dY_m - dX^m B_{mn}) e^{na}, \\ \tilde{\Pi}^{a'} &= dX^m e_m^{a'} - (dY_m - dX^m B_{mn}) e^{na'}. \end{aligned}$$

In order to reduce the number of degrees of freedom to half one imposes duality constraints

$$\star \Pi^a = \Pi^a,$$

$$\star \tilde{\Pi}^{a'} = -\Pi^{a'}$$

(which may also be written as $\star dZ^M = L^{MN} G_{NP} dZ^P$). So Π and $\tilde{\Pi}$ are left- and right-moving, respectively. The (anti-)selfduality implies that the equations of motion are automatically satisfied.

Quantum calculations need to use holomorphic factorisation. The partition function has been shown to agree with string theory [14].

We would now like to do something similar for branes of M-theory. There are several issues. Branes with $p > 1$ are nonlinear. There is no conformal gauge. If there is a duality relation involved, dualisation with which world-volume metric? There is essentially one candidate: the pullback Γ_{ij} of the metric G_{MN} on the extended space determined by the $E_{n(n)}/K(E_{n(n)})$ vielbein.

Dualisation of some 1-form Π^A on a $(p+1)$ -dimensional brane gives a p -form. One needs some invariant tensor c to be able to write

$$\star_{\Gamma} \Pi^A = c^A_{A_1 \dots A_p} \Pi^{A_1} \wedge \dots \wedge \Pi^{A_p} .$$

Which are the possible invariant tensors? Let us make a list. We assume that the tensor $c_{AA_1 \dots A_p}$ is totally antisymmetric, and look for singlets under $K(E_{n(n)})$ in the $p+1$ -fold product $\wedge^{p+1} R$ of the coordinate representation. (The table is not complete.)

n	$R_{E_{n(n)}}$	$R_{K(E_{n(n)})}$	repr.	$p =$	2	3	4	5	6	7	8	9	10
4	10	10			×				×				
5	16	(4, 4)				×				×			
6	27	27					×				×		
7	56	$28 \oplus \bar{28}$						×				×	
8	248	$120 \oplus 128$							×	...	?		

Table 3. Series of $K(E_{n(n)})$ -invariant antisymmetric tensors.

In order to show that the idea works, we will do one example in detail, the case of an M2 instanton with $n = 4$. The term “instanton” just means that we consider a euclidean membrane spanning *only* internal directions. This simplifies things: we don’t need to worry how the movement of the membrane in the uncompactified directions enter into the world-volume metric. (This has to be investigated, of course.)

The U-duality group is $SL(5)$. Coordinates are $Z^{\mathcal{M}} = Z^{MN} = (X^m, Y^{mn})$ where $M = 1, \dots, 5$, $m = 1, \dots, 4$. The metric on the repr. 10 is $G_{MN, PQ} = \frac{1}{2} G_{M[P} G_{Q]N}$, where G_{MN} is the metric on the repr 5. The $SL(5)$ vielbein is parametrised as

$$E_M^A = \begin{bmatrix} e^{1/3} & -e^{1/3} C^n e_n^a \\ 0 & e^{-1/3} e_m^a \end{bmatrix} ,$$

where $C^m = \frac{1}{6}\varepsilon^{mnpq}C_{npq}$. This gives the metric

$$G_{MN} = \begin{bmatrix} g^{1/3}(1 + C^p C^q g_{pq}) & -C^p g_{pn} \\ -g_{mp} C^p & g^{-1/3} g_{mn} \end{bmatrix} .$$

The induced metric on the world-volume is $\Gamma_{ij} = \frac{1}{2}\partial_i Z^{MN}\partial_j Z^{PQ}G_{MP}G_{NQ}$. Splitting Z^{MN} into $Z^{0m} = X^m$ and $Z^{mn} = g^{1/3}Y^{mn}$, one gets

$$\begin{aligned} \Gamma_{ij} &= \frac{1}{2}\partial_i Z^{mn}\partial_j dZ^{pq}G_{mp}G_{nq} + 2\partial_{(i}Z^{0n}\partial_{j)}Z^{pq}G_{0p}G_{nq} \\ &\quad + \partial_i Z^{0m}\partial_j Z^{0n}(G_{00}G_{mn} - G_{0m}G_{0n}) \\ &= \frac{1}{2}\partial_i Y^{mn}\partial_j Y^{pq}g_{mp}g_{nq} - 2\partial_{(i}X^m\partial_{j)}Y^{np}C_n g_{mp} \\ &\quad + \partial_i X^m\partial_j X^n((1 + C^2)g_{mn} - C_m C_n) . \end{aligned}$$

We want to use this metric in the dualisation, and see if we can find reasonable solutions of Y in terms of X to the duality relation

$$\star_\Gamma dZ^{MN} = \alpha dZ^{MP} \wedge dZ^{NQ} G_{PQ} .$$

An Ansatz for the solution may be

$$\star_\gamma dY^{mn} = a dX^m \wedge dX^n + b \star_\gamma dX^{[m} C^{n]} ,$$

where dualisation is with γ_{ij} , the pullback to T^4 of g_{mn} . Inserting this Ansatz into the metric gives

$$\Gamma_{ij} = (1 + a^2 + (1 + \frac{1}{2}b)^2 C^2)\gamma_{ij} - (1 + \frac{1}{2}b)^2 C_i C_j .$$

We choose $b = -2$, otherwise the ordinary membrane eqs. of motion can not be recovered. Then $\Gamma_{ij} = (1 + a^2)\gamma_{ij}$, and acting on a 1-form, $\star_\Gamma = \sqrt{1 + a^2}\star_\gamma$. Now we insert this into the duality relations, which then read

$$\begin{aligned} \alpha^{-1}\sqrt{1 + a^2}\star_\gamma dY^{mn} &= dY^{mp} \wedge dY^n{}_p + 2dX^{[m} \wedge dY^{n]p} C_p + (1 + C^2)dX^m \wedge dX^n , \\ -\alpha^{-1}\sqrt{1 + a^2}\star_\gamma dX^m &= dY^{mp} \wedge dX_p + dX^m \wedge dX^p C_p . \end{aligned}$$

The two equations must be consistent with each other and with the Ansatz, which gives two equations,

$$\alpha = \frac{\sqrt{1 + a^2}}{2a} = \frac{a}{\sqrt{1 + a^2}} ,$$

with the solutions $a = \pm 1$, $\alpha = \pm \frac{1}{\sqrt{2}}$. This is quite nontrivial, and depends on several cancellations. It shows that the duality relation provides a U-duality covariant description of the membrane on T^4 .

Note that the duality relation implies that the equations of motion following from

$$S = \int d^3\xi \sqrt{\Gamma}$$

are satisfied. There is, and should not be, a separate WZ term, since the C -field is contained in the metric on the enlarged space.

For the future, one should consider branes moving also in the uncompactified directions, with couplings to all background fields, and also different values of n . Supersymmetrisation is presumably straightforward.

We would like to make some comments about branes with vector or tensor fields. How do world-volume vector or tensor fields, such as the vector potential on D-branes, or the self-dual 2-form on the M5-brane, arise? Normally such fields are identified with the value of a background tensor field on the brane, but these fields are now unified with the metric.

If n is large enough, a U-duality rotation relates *e.g.* an M2-brane and an M5-brane. Either the tensor field is not present in the dynamics on the extended space, but arises as a parametrisation of the orientation of the brane in the extra directions, or it is made to disappear for the M2 interpretation.

No concrete case has been worked through, and we will instead take D-branes as an example, where a qualitative description has been given by Hull [13]. All D-branes on the doubled torus are n -dimensional and span a light-like n -plane with respect to the metric L . Which brane is “seen” depends on the choice of polarisation, *i.e.*, of the choice of the embedding of $GL(n)$ in $SO(n, n)$.

Consider an orientation where k of the directions lie in the X directions (*i.e.*, the rank of $\partial_i X^m$ is k). Then there are $n - k$ transverse directions in X and k in Y . The extra (Y) transverse directions are related to the vector field. A bit more precisely: The possible duality relations one can write down for an n -dimensional brane are

$$\begin{aligned}\Pi^a &= \pm \frac{1}{(n-1)!} \varepsilon^a_{a_2 \dots a_n} \star_\Gamma (\Pi^{a_2} \wedge \dots \wedge \Pi^{a_n}) , \\ \tilde{\Pi}^{a'} &= \pm \frac{1}{(n-1)!} \varepsilon^{a'}_{a'_2 \dots a'_n} \star_\Gamma (\tilde{\Pi}^{a'_2} \wedge \dots \wedge \tilde{\Pi}^{a'_n}) .\end{aligned}$$

Take the case where the D-brane fills the X -space. Then $\partial_i X^m$ is non-degenerate, and one may try

$$\begin{aligned}\partial_i Y_m &\sim f_{ij} \varepsilon^{jj_2 \dots j_n} \varepsilon_{mm_2 \dots m_n} \partial_{j_2} X^{m_2} \dots \partial_{j_n} X^{m_n} + \partial_i X^n B_{nm} \\ &\sim f_{ij} (\partial X)^{-1j}_m + \partial_i X^n B_{nm} .\end{aligned}$$

Suppressing the matrices ∂X (*i.e.*, using a static gauge where it is the unit matrix), one has

$$\begin{aligned}\Pi &= 1 + f , \\ \tilde{\Pi} &= 1 - f ,\end{aligned}$$

and this is a (quite nontrivial) solution to the proposed duality relations. It is also the correct deformation preserving the lightlikeness of the orientation of the brane. The dynamics of the vector field (eqs. of motion and Bianchi identities) should be examined

closer. Here it arises in the solution of the generalised self-duality constraint. It is still not excluded that a complete formulation demands a field that lives intrinsically on the brane. The corresponding investigation should also be carried on to M-branes in cases where $n \geq 6$.

Let us summarise, by giving an outlook, ranging from technical points to wild speculation:

There are many cases to be worked through in detail, especially the ones with world-volume tensor fields. Coupling to all background fields has to be included, but should be straightforward. Supersymmetry and κ -symmetry should certainly be manageable, but will take some work. It may be interesting to examine the special cycles on the internal manifold from the viewpoint of supersymmetric calibrations.

Polarisations—choices of the physical subspace—correspond to embeddings of $SL(n)$ into $E_{n(n)}$. Pure spinors parametrise the corresponding cosets in the doubled formalism in the context of T-duality (and play an important rôle in generalised complex geometry). Is there a natural generalisations of pure spinors to the M-theoretic setting? (The answer is probably yes.)

A question that has bearing on the E_{11} proposal, concerns the “reality” of the extra coordinates. Is there a Borisov–Ogievetsky-like construction [15] containing diffeomorphisms of the torus and gauge transformations of tensor fields, without generating “everything”? If this is possible, what is the relation to higher spin theory? Like higher spin theory, this would be a model with massless higher spin fields, obtained by extending the coordinates with some tensorial objects.

It should be noted that descriptions of brane dynamics respecting global symmetries of string theory or M-theory have been worked out earlier with other methods (see *e.g.* refs. [16, 17, 18, 19, 20]). We expect such descriptions to be reproduced by the present program when branes spanning only the uncompactified direction are considered.

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